Time-domain decoupled poroelastic equations for saturated porous media

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Introduction

- Earthquakes cause a variety of hydrological phenomena, e.g. groundwater level fluctuations and oil production variations
- *Poroelasticity* has been extensively utilized to explain and model these phenomena

Poroelasticity

- *Poroelasticity* refers to the strong time-dependent coupling between applied stress and pore fluid pressure that is often encountered in hydrogeological problems involving groundwater aquifers, subsurface waste disposal sites, and oil reservoirs

Biot Theory

- The mathematical description of poroelasticity in a homogeneous, isotropic, elastic porous medium containing a single compressible viscous fluid was pioneered in a series of celebrated papers by Biot [1941, 1956, 1962]

Motivation

- However, closed-form analytical solutions of the Biot equations subject to a variety of initial and boundary conditions are very limited
- To obtain these solutions, it is essential to perform a *normal coordinate transformation* that decouples them

Biot Model Equations

$$G\nabla^2\vec{u}_s + (H-G)\vec{\nabla}e - C\vec{\nabla}\zeta = \rho\frac{\partial^2\vec{u}_s}{\partial t^2} + \rho_f\frac{\partial^2\vec{w}}{\partial t^2}$$

$$C\vec{\nabla}e - M\vec{\nabla}\zeta = \rho_f\frac{\partial^2\vec{u}_s}{\partial t^2} + m\frac{\partial^2\vec{w}}{\partial t^2} + \frac{b}{\phi^2}\frac{\partial\vec{w}}{\partial t}$$

where $\vec{u}_\alpha$ signifies the displacement vector for two immiscible continua:

fluid phase ($\alpha = f$) and solid phase ($\alpha = s$)

$\vec{w} = \phi(\vec{u}_f - \vec{u}_s)$ denotes the displacement vector of the fluid relative to the solid phase, ϕ being porosity

$\zeta = -\vec{\nabla}\cdot\vec{w}$ is the increment of fluid content, representing the accumulation or depletion of pore fluid in a volume element attached to the solid framework

$e = \vec{\nabla}\cdot\vec{u}_s$ expresses the dilatation of the solid phase

ρ_α and θ_α are the mass density and the volume fraction of phase

H, C, M, and G are elasticity moduli, , $\rho = \sum\rho_\alpha\theta_\alpha$

Inertial Coupling

The quantity m is a parameter proposed by Biot to account for inertial coupling between the solid and fluid phases

$$m = \frac{\alpha_s \rho_f}{\phi}$$

where α_s is a tortuosity factor, a quantitative measure of how much the pore orientational structure restricts fluid flow, leading to inertial coupling

Viscous Coupling

Assume that Poiseuille flow is valid for describing dissipation caused by the relative motions of solid and fluid

$$b = \frac{\mu_f \phi^2}{k_s}$$

where μ_f is the pore fluid dynamic shear viscosity

k_s is the intrinsic permeability of the porous medium

Dilatational Wave Equations

$$H\nabla^2 e - C\nabla^2 \zeta = \rho \frac{\partial^2 e}{\partial t^2} - \rho_f \frac{\partial^2 \zeta}{\partial t^2}$$

$$C\nabla^2 e - M\nabla^2 \zeta = \rho_f \frac{\partial^2 e}{\partial t^2} - \frac{\alpha_s \rho_f}{\phi} \frac{\partial^2 \zeta}{\partial t^2} - \frac{\mu_f}{k_s} \frac{\partial \zeta}{\partial t}$$

Helmholtz equations

Fourier Transformation

$$e(\vec{x},t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{e}(\vec{x},\omega)\exp(-i\omega t)d\omega$$

$$\zeta(\vec{x},t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\zeta}(\vec{x},\omega)\exp(-i\omega t)d\omega$$

$$[\nabla^2 + \lambda_{\pm}(\omega)]\Phi_{\pm}(\vec{x},\omega) = 0$$ Berryman [1983]

eigenvalues: $\lambda_{\pm}(\omega) = \frac{1}{2}\{(\bar{b}+\bar{f}) \pm [(\bar{b}-\bar{f})^2 + 4\bar{c}\bar{d}]^{\frac{1}{2}}\}$

eigenvectors: $\Phi_{\pm}(\vec{x},\omega) = \Gamma_{\pm}\tilde{e} - \tilde{\zeta}$

Parameters

$$\Gamma_{\pm} = \bar{d}(\lambda_{\pm} - \bar{b})^{-1} = (\lambda_{\pm} - \bar{f})\bar{c}^{-1} = \frac{1}{2\bar{c}}\{(\bar{b} - \bar{f}) \pm [(\bar{b} - \bar{f})^2 + 4\bar{c}\bar{d}]^{\frac{1}{2}}\}$$

$$\bar{b} = \omega^2(\rho M - \rho_f C)\Delta^{-1}$$

$$\bar{c} = \omega^2(\rho_f M - qC)\Delta^{-1}$$

$$\bar{d} = \omega^2(\rho_f H - \rho C)\Delta^{-1}$$

$$\bar{f} = \omega^2(qH - \rho_f C)\Delta^{-1}$$

$$\Delta = MH - C^2$$

$$q = \frac{\alpha_s \rho_f}{\phi} + \frac{i\mu_f}{\omega k_s}$$

MacLaurin expansions

To examine dilatational wave equations at low values of ω, we employ MacLaurin expansions of the coefficients $\Gamma_{\pm}$

$$\Gamma_{+} \approx \frac{H}{C} + i\frac{\Delta}{C^2}\left(\frac{C}{H}\frac{\rho}{\rho_f} - 1\right)\rho_f \frac{k_s}{\mu_f}\omega + \mathrm{O}(\omega^2)$$

$$\Gamma_{-} \approx -i\left(\frac{C}{H}\frac{\rho}{\rho_f} - 1\right)\rho_f \frac{k_s}{\mu_f}\omega + \mathrm{O}(\omega^2)$$

Lowest order in ω

To lowest order in ω

$$\Gamma_{+} \approx \frac{H}{C}, \qquad \Gamma_{-} \approx 0$$

It follows that, to the same order in ω, the eigenvalues $\lambda_{\pm}(\omega)$ are

$$\lambda_{+} = \bar{d}\Gamma_{+}^{-1} + \bar{b} \approx \frac{\bar{d}C}{H} + \bar{b} = \frac{\omega^2}{\Delta}[\rho_f C - \rho\frac{C^2}{H} + \rho M - \rho_f C] = \frac{\rho}{H}\omega^2$$

$$\lambda_{-} = \bar{c}\Gamma_{-} + \bar{f} \approx \bar{f} = \frac{1}{\Delta}(\frac{\alpha_s \rho_f}{\phi}H - \rho_f C)\omega^2 + \frac{i}{\Delta}\frac{\mu_f}{k_s}H\omega$$

Time-domain Decoupled Equations [Lo *et al.*, 2005]

Wave Equation ← ***Biot Fast Wave !***

$$\frac{\partial^2}{\partial t^2}\left(\frac{H}{C}e-\zeta\right)=\frac{H}{\rho}\nabla^2\left(\frac{H}{C}e-\zeta\right)$$

Telegraph Equation ← ***Biot Slow Wave !***

$$\frac{\partial^2\zeta}{\partial t^2}+\frac{H}{(mH-\rho_f C)}\frac{\mu_f}{k_s}\frac{\partial\zeta}{\partial t}=\frac{(MH-C^2)}{(mH-\rho_f C)}\nabla^2\zeta$$

Linear Stress-Strain Relations

$$\overline{\overline{\sigma}} = 2G\overline{\overline{e}} + [(H-2G)e - C\zeta]\overline{\overline{\delta}}$$

$$p_f = -Ce + M\zeta$$

where $\overline{\overline{\sigma}}$ is the total stress tensor

p_f is the pore pressure

The total dilatational stress is defined

$$\sigma_{kk} = \sigma^{xx} + \sigma^{yy} + \sigma^{zz} = (3H-4G)e - 3C\zeta$$

Time-domain Decoupled Equations [Lo *et al.*, 2005]

Wave Equation

$$\frac{\partial^2}{\partial t^2}(a_1\sigma_{kk}+a_2p_f)=\frac{H}{\rho}\nabla^2(a_1\sigma_{kk}+a_2p_f)$$

Telegraph Equation

$$\frac{\partial^2(a_3\sigma_{kk}+a_4p_f)}{\partial t^2}+\frac{H}{(mH-\rho_fC)}\frac{\mu_f}{k_s}\frac{\partial(a_3\sigma_{kk}+a_4p_f)}{\partial t}=\frac{(MH-C^2)}{(mH-\rho_fC)}\nabla^2(a_3\sigma_{kk}+a_4p_f)$$

where $a_1=\dfrac{MH-C^2}{C(3MH-4MG-3C^2)}$; $\quad a_2=\dfrac{4G}{(3MH-4MG-3C^2)}$

$a_3=\dfrac{C}{(3MH-4MG-3C^2)}$; $\quad a_4=\dfrac{(3H-4G)}{(3MH-4MG-3C^2)}$

Pore fluid pressure can be determined analytically!

Conditions for “low frequency”

The validity of our eqautions requires that

$$\rho_f \frac{k_s}{\mu_f} \omega \ll 1 \qquad \Rightarrow \qquad \omega \ll \frac{\mu_f}{\rho_f k_s} = \omega_c$$

Intrinsic time scale exists in Biot theory

$$\tau = \frac{\rho_f k_s}{\mu_f}$$

Values of the Critical Frequency (kHz)

[Lo *et al.*, 2005]	Water	TCE	Oil
Unconsolidated sand	9.118	3.476	17.179
Massilon sandstone	1114.454	424.924	2099.737
Lost Hills diatomite	3343.363	1274.774	6299.212

>> well above the seismic frequency range!

Conclusions

- A propagating wave equation and a dissipative wave equation for the Biot fast and slow compressional waves, respectively, were derived in the low-frequency limit

Conclusions

- The dependent variables of our decoupled equations can be expressed in terms of *linear combinations of solid dilatation and linearized increment of fluid content*, alternatively, *linear combinations of pore fluid pressure and total dilatational stress*

Conclusions

- The precise definition of “low frequency” is equivalent to requiring the (angular) frequency of wave excitation to be much smaller than a critical frequency equal to kinematic viscosity of the pore fluid divided by the permeability of the porous medium

Conclusions

- Illustrative numerical calculations performed for porous media containing water or NAPL indicate that the critical frequency will lie typically in the kHz to MHz range, well above the seismic frequency range